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Classical Duality Symmetries in Two Dimensions [1]

John H. Schwarz[2]

California Institute of Technology, Pasadena, CA 91125, USA

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Abstract

Many two-dimensional classical field theories have hidden symmetries that form an infinite-dimensional algebra. For those examples that correspond to effective descriptions of compactified superstring theories, the duality group is expected to be a large discrete subgroup of the hidden symmetry group. With this motivation, we explore the hidden symmetries of principal chiral models and symmetric space models.

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[2]Email: jhs@theory.caltech.edu

1 Introduction

Despite the impressive progress that has been achieved in understanding string theory during the past decade, the theory has not yet been satisfactorily formulated. There are rules for identifying classical solutions, but we do not know the equation that these "solutions" solve. There are also rules for how to compute quantum corrections to these classical solutions to any finite order in perturbation theory, but we do not know how to compute non-perturbative quantum effects. Recently, a window onto non-perturbative string theory has begun to open with the discovery of various duality symmetries and mappings that can relate weak coupling and strong coupling[1]. It seems possible that by developing a deeper understanding of these dualities we will eventually be led to the long-sought non-perturbative formulation of the theory. More specifically, there is ever increasing evidence that at a fundamental level there is just one superstring theory[2]. It seems reasonable that it should be largely characterized by its group of symmetries. This group should contain all the duality symmetries, which are to be viewed as gauge symmetries in the sense that they relate equivalent configurations that should be counted only once in the path integral that defines the theory.

The specific duality groups that have been identified to date should be viewed as subgroups of the complete duality group of string theory. The point is that, with currently available techniques, it is only possible to study superstrings in specific classical backgrounds, for example ones in which some spatial dimensions are compactified on a particular manifold. For any such choice, some of the duality symmetries are easily identified and others are completely hidden. The "visible" ones are those that can be understood in terms of the massless modes whereas the rest must involve massive modes in a complicated way. In particular, the visible group in question is realized nonlinearly on the massless scalars. The effective classical theory has a continuous symmetry group, but string and quantum effects restrict it to a discrete duality group. Thus, for example, the heterotic string toroidally compactified to four dimensions has an $O(6,22;\mathbb{Z})$ T-duality group and an $SL(2,\mathbb{Z})$ S-duality group. In three dimensions these are combined and extended to give an $O(8,24;\mathbb{Z})$ U-duality group[3]. These examples illustrate an important point: the more dimensions are compactified, the more duality symmetries become visible. Since my goal is to understand the complete group, this suggests that it should be worthwhile to consider cases in lower dimensions in which even larger symmetry groups can be identified. In the case of toroidally compactified type II superstrings in d dimensions, the duality groups appear to be integral subgroups of maximally noncompact forms of E_{11-d}. If this rule extrapolates correctly, it would give E_9 (the affine extension of E_8) in two dimensions and E_{10} (a hyperbolic Lie algebra) in one dimension[4]. It may be that affine Lie algebras are generic in two dimensions and hyperbolic ones are generic in one dimension. As a

modest first step to see if this is the case, I have investigated the affine symmetry algebras of certain classes of two dimensional theories, principal chiral models and symmetric space models. The principal chiral models are a warm-up exercise, whereas symmetric space models are relevant to our problem. However, to further simplify matters I have omitted gravity, fermions, and quantum effects. The results presented here summarize a recent paper to which I refer the reader for additional details[5].

In Ref. [5], I attempted to sketch the history of the study of hidden symmetries in two-dimensional models. Here I will simply remark that relativists, beginning with Geroch[6] in 1971, studied hidden symmetries of classical theories coupled to gravity in two dimensions, which corresponds to four-dimensional Einstein theory with two commuting Killing vectors. The symmetries enable one to construct new solutions out of old ones. Field theorists, on the other hand, studied two-dimensional quantum theories (mostly without gravity) as simpler analogs of four-dimensional gauge theories. Hidden non-local conserved charges of principal chiral models were discovered by Pohlmeyer and Lüscher[7]. The algebra of the corresponding symmetry transformations was studied by Dolan, Wu, and others[8]. Hidden symmetries in the supergravity context have been explored most notably by Julia, Breitenlohner, Maison, and Nicolai[9, 10, 11]. Preliminary studies of these symmetries for 2D string theory have been given by Bakas and Maharana[12]. A more detailed discussion for the toroidally compactified heterotic string has been given by Sen[13].

2 Principal Chiral Models

Principal chiral models (PCM's) are based on fields $g(x)$ that map space-time into a group manifold G, which we may assume to be compact. Even though these models are not directly relevant to the string theory and supergravity applications that we have in mind, they serve as a good warm-up exercise, as well as being of some interest in their own right. Symmetric space models, which are relevant, share many of the same features, but are a little more complicated. They will be described in the next section.

The classical theory of PCM's, in any dimension, is defined by the lagrangian

$$\mathcal{L} = \eta^{\mu\nu} tr(A_\mu A_\nu), \tag{1}$$

where the connection A_μ is defined in terms of the group variables by

$$A_\mu = g^{-1}\partial_\mu g = \sum A^i_\mu T_i. \tag{2}$$

Here $\eta^{\mu\nu}$ denotes the Minkowski metric for flat space-time, and the T_i are the generators of the Lie algebra,

$$[T_i, T_j] = f_{ij}{}^k T_k. \tag{3}$$

They may be taken to be matrices in any convenient representation. The classical equation of motion is derived by letting δg be an arbitrary infinitesimal variation of g for which $\eta = g^{-1}\delta g$ belongs to the Lie algebra $\mathcal{G}$. Under this variation

$$\delta A_\mu = D_\mu \eta = \partial_\mu \eta + [A_\mu, \eta], \tag{4}$$

and thus

$$\delta \mathcal{L} = 2\,\mathrm{tr}(A^\mu D_\mu \eta) = 2\,\mathrm{tr}(A^\mu \partial_\mu \eta). \tag{5}$$

From this it follows that the classical equation of motion is

$$\partial_\mu A^\mu = 0, \tag{6}$$

as is well-known. Since A_μ is pure gauge, the Bianchi identity is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \tag{7}$$

The PCM in any dimension has manifest global $G \times G$ symmetry corresponding to left and right group multiplication. Remarkably, in two dimensions this is just a small subgroup of a much larger group of "hidden" symmetries. To describe how they arise, it is convenient to introduce light-cone coordinates

$$x^\pm = x^0 \pm x^1, \quad \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1). \tag{8}$$

Expressed in terms of these coordinates, the equation of motion and Bianchi identity take the forms

$$\partial_\mu A^\mu = \partial_+ A_- + \partial_- A_+ = 0 \tag{9}$$

$$F_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0. \tag{10}$$

A standard technique (sometimes called the "inverse scattering method") for discovering the "hidden symmetries" of integrable models, such as a PCM in two dimensions, begins by considering a pair of linear differential equations, known as a Lax pair. In the present context the appropriate equations are

$$(\partial_+ + \alpha_+ A_+)X = 0 \quad \text{and} \quad (\partial_- + \alpha_- A_-)X = 0, \tag{11}$$

where $\alpha_\pm$ are constants. These equations are compatible, as a consequence of equations (9) and (10), provided that

$$\alpha_+ + \alpha_- = 2\alpha_+\alpha_-. \tag{12}$$

It is convenient to write the solutions to this equation in terms of a "spectral parameter" t in the form

$$\alpha_+ = \frac{t}{t-1}, \quad \alpha_- = \frac{t}{t+1}. \tag{13}$$

The variable X in eqs. (11) is a group-valued function of the space-time coordinate, as well as the spectral parameter. The integration constant can be fixed by requiring that X reduces to the identity element of the group at a "base point" x_0^μ. A formal solution to eqs. (11) is then given by a path-ordered exponential

$$X(x,t) = P\exp\Big\{-\int_{x_0}^{x}(\alpha_+ A_+ dy^+ + \alpha_- A_- dy^-)\Big\}, \tag{14}$$

where the path ordering has x on the left and x_0 on the right. The integral is independent of the contour provided the space-time is simply connected. This is the case, since we are assuming a flat Minkowski space-time. If one were to choose a circular spatial dimension instead, the multivaluedness of X would raise new issues, which we will not consider here. Note that X is group-valued for any real t, except for the singular points $t = \pm 1$.

The next step is to consider the variation $g^{-1}\delta g = \eta$, with

$$\eta(\epsilon, t) = X(t)\epsilon X(t)^{-1}, \tag{15}$$

where $\epsilon = \sum \epsilon^i T_i$ and ϵ^i are infinitesimal constants. The claim is that the variation $\delta(\epsilon,t)g = g\eta$ preserves the equation of motion $\partial \cdot A = 0$ and, therefore, describes symmetries of the classical theory. To show this, one simply notes that the Lax pair implies that

$$\delta A_\pm = D_\pm \eta = \partial_\pm \eta + [A_\pm, \eta] = \pm\frac{1}{t}\partial_\pm \eta, \tag{16}$$

and, therefore, $\partial \cdot (\delta A) = 0$ as required.

Let us now consider the commutator of two symmetry transformations $[\delta(\epsilon_1,t_1),\delta(\epsilon_2,t_2)]$. The key identity that is required is

$$\delta(\epsilon_1,t_1)X(t_2) = \frac{t_2}{t_1 - t_2}(\eta(\epsilon_1,t_1)X(t_2) - X(t_2)\epsilon_1). \tag{17}$$

Identities such as this are used frequently in this work. The method of proof is always the same. One shows that both sides of the equation satisfy the same pair of linear differential equations and boundary conditions and then concludes by uniqueness that they must be equal. The required equations are obtained by varying the Lax pair. Using this identity it is easy to derive

$$[\delta(\epsilon_1,t_1),\delta(\epsilon_2,t_2)] = \frac{t_1\delta(\epsilon_{12},t_1) - t_2\delta(\epsilon_{12},t_2)}{t_1 - t_2}, \tag{18}$$

where

$$\epsilon_{12} = [\epsilon_1, \epsilon_2] = f_{ij}{}^k \epsilon_1^i \epsilon_2^j T_k. \tag{19}$$

In order to understand the relationship between the algebra (18) and current algebra associated with the group G, we need to do some sort of mode expansion with respect to the parameter t. The standard approach in the literature is to do a power series expansion in t, $\delta(\epsilon, t) = \sum_{n=0}^{\infty} \delta_n(\epsilon) t^n$, identifying the $\delta_n(\epsilon)$ as distinct symmetry transformations. This gives half of a current algebra:

$$[\delta_m(\epsilon_1), \delta_n(\epsilon_2)] = \delta_{m+n}(\epsilon_{12}) \quad m, n \geq 0. \tag{20}$$

Actually, $\delta(\epsilon, t)$ contains more information than is extracted in this way, and in Ref. [5] I found a nice way to reveal it. The idea is to define variations $\Delta_n(\epsilon) g$ for all integers n by the contour integral

$$\Delta_n(\epsilon) g = \int_{\mathcal{C}} \frac{dt}{2\pi i} t^{-n-1} \delta(\epsilon, t) g \quad n \in \mathbb{Z}, \tag{21}$$

where the contour $\mathcal{C} = \mathcal{C}_+ + \mathcal{C}_-$ and $\mathcal{C}_\pm$ are small clockwise circles about $t = \pm 1$. By distorting contours it is easy to show that $\Delta_n(\epsilon) = \delta_n(\epsilon)$ for $n > 0$. The negative integers n are given entirely by poles at $t = \infty$. In other words, they correspond to the coefficients in a series expansion in inverse powers of t. Δ_0 receives contributions from poles at both $t = 0$ and $t = \infty$. (Explicitly, $\Delta_0(\epsilon) g = [g, \epsilon]$.) Because $g^{-1} \Delta_n g$ can be related to such series expansions, it is clear that it is Lie-algebra valued.[3]

Using the definition (21) and the commutator (18), it is an easy application of Cauchy's theorem to deduce the affine current algebra (without center)

$$[\Delta_m(\epsilon_1), \Delta_n(\epsilon_2)] = \Delta_{m+n}(\epsilon_{12}) \qquad m, n \in \mathbb{Z}. \tag{22}$$

Equivalently, in terms of charges we have

$$[J_m^i, J_n^j] = f^{ij}{}_k J_{m+n}^k. \tag{23}$$

Having found current algebra symmetries for classical PCM's, it is plausible that they should also have Virasoro symmetries[14]. We now show that, modulo an interesting detail, this is indeed the case. Since the infinitesimal parameter in this case is not Lie-algebra valued, it can be omitted without ambiguity. With this understanding, the Virasoro-like transformation

$$\delta^V(t) g = g((t^2 - 1)\dot{X}(t) X(t)^{-1} + I), \tag{24}$$

[3] If one tried to define further symmetries corresponding to the contours $\mathcal{C}_\pm$ separately or by allowing n to be non-integer, the transformations defined in this way would also appear to preserve $\partial \cdot A = 0$. However, these could fail to be honest symmetries because $g^{-1} \delta g$ might not be Lie-algebra valued.

where the dot denotes a t derivative and

$$I = \dot{X}(0) = \int_{x_0}^{x} (A_+ dy^+ - A_- dy^-), \tag{25}$$

can be shown to be an invariance of the equation of motion $\partial \cdot A = 0$. We can extract modes δ_n^V, for all integers n, by the same contour integral definition used above

$$\delta_n^V g = \int_{\mathcal{C}} \frac{dt}{2\pi i} t^{-n-1} \delta^V(t) g. \tag{26}$$

Again, contour deformations give pole contributions at $t = 0$ and $t = \infty$ only, and therefore, one sees that $g^{-1}\delta_n^V g$ is Lie-algebra valued.

The analysis of the algebra proceeds in the same way as for the current algebra, though the formulas are quite a bit more complicated. For example, commuting a Virasoro symmetry transformation with a current algebra symmetry transformation gives

$$[\delta^V(t_1), \delta(\epsilon, t_2)]g = \Big(\frac{1}{t_2}(\delta(\epsilon, 0) - \delta(\epsilon, t_2))$$
$$+ \frac{t_2(t_1^2 - 1)}{(t_1 - t_2)^2}(\delta(\epsilon, t_1) - \delta(\epsilon, t_2)) + \frac{t_1(1 - t_2^2)}{t_1 - t_2} \ \frac{\partial}{\partial t_2} \delta(\epsilon, t_2)\Big) g. \tag{27}$$

Using eq. (27) and the contour integral definitions, one finds after some integrations by parts and use of Cauchy's theorem that

$$[\delta_m^V, \Delta_n(\epsilon)]g = n \int_{\mathcal{C}} \frac{dt}{2\pi i} t^{-m-n-2}(t^2 - 1)\delta(\epsilon, t) g. \tag{28}$$

Now let us re-express the algebra in terms of charges J_n^i (as before) and K_m (corresponding to δ_m^V). In this notation, eq. (28) becomes

$$[K_m, J_n^i] = n(J_{m+n-1}^i - J_{m+n+1}^i). \tag{29}$$

This formula is to be contrasted with what one would expect for the usual Virasoro generators L_n

$$[L_m, J_n^i] = -nJ_{m+n}^i. \tag{30}$$

Comparing equations, we see that we can make contact with the usual (centerless) Virasoro algebra if we identify

$$K_n = L_{n+1} - L_{n-1}. \tag{31}$$

However, it should be stressed that we have only defined the differences K_n and not the individual L_n's. Still, this identification is useful since it tells us that

$$[K_m, K_n] = (m - n)(K_{m+n+1} - K_{m+n-1}). \tag{32}$$

Let us see what happens if we try to construct the L_n's. The easiest approach is to define $K(\sigma) = \sum_{-\infty}^{\infty} K_n e^{in\sigma}$ and $T(\sigma) = \sum_{-\infty}^{\infty} L_n e^{in\sigma}$. Then eq. (31) implies that

$$T(\sigma) = \frac{i}{2} \frac{K(\sigma)}{\sin \sigma}. \tag{33}$$

The remarkable fact is that $K(\sigma)$ does not vanish at $\sigma = 0$ and $\sigma = \pi$. Therefore, $T(\sigma)$ diverges at these points and the individual L_n's do not exist. The integrals that would define them are logarithmically divergent.

3 Symmetric Space Models

An interesting class of integrable two-dimensional models consists of theories whose fields map the space-time into a symmetric space. Let G be a simple group and H a subgroup of G. Then the Lie algebra $\mathcal{G}$ can be decomposed into the Lie algebra $\mathcal{H}$ and its orthogonal complement $\mathcal{K}$, which contains the generators of the coset G/H. The coset space G/H is called a symmetric space if $[\mathcal{K}, \mathcal{K}] \subset \mathcal{H}$, in other words the commutators of elements of $\mathcal{K}$ belong to $\mathcal{H}$. The examples that arise in string theory and supergravity are non-compact symmetric space models (SSM's), such as those mentioned in the introduction. For such models, G is a non-compact Lie group and H is its maximal compact subgroup. The generators of $\mathcal{H}$ are antihermitian and those of $\mathcal{K}$ are hermitian. Therefore, since the commutator of two hermitian matrices is antihermitian, $[\mathcal{K}, \mathcal{K}] \subset \mathcal{H}$ and G/H is a (non-compact) symmetric space.

Symmetric space models can be formulated starting with arbitrary G-valued fields, $g(x)$, like those of PCM's. To construct an SSM, we associate local H symmetry with left multiplication and global G symmetry with right multiplication. Thus, we require invariance under infinitesimal transformations of the form

$$\delta g = -h(x) g + g \epsilon \quad h \in \mathcal{H}, \ \epsilon \in \mathcal{G}. \tag{34}$$

The local symmetry effectively removes H degrees of freedom so that only those of the coset remain. The next step is to define

$$P_\mu = \frac{1}{2}(g \partial_\mu g^{-1} + \partial_\mu g^{-1\dagger} g^\dagger), \tag{35}$$

$$A_\mu = -2 g^{-1} P_\mu g. \tag{36}$$

and to observe that this A_μ is invariant under local H transformations. It can be recast in an alternative form that makes this manifest, specifically

$$A_\mu = M^{-1} \partial_\mu M, \tag{37}$$

where

$$M = g^\dagger g. \tag{38}$$

Note that g and M are analogous to a vielbein and metric in general relativity. M parametrizes the symmetric space G/H without extra degrees of freedom. In the case of a compact SSM the factor $g^\dagger$ in the definition of M must be generalized to a quantity $\tilde{g}$, which is described in Ref. [5]. Since A_μ is pure gauge, its field strength vanishes

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \tag{39}$$

The lagrangian is $\mathcal{L} = \mathrm{tr}(A^\mu A_\mu)$ and the classical equation of motion is

$$\partial^\mu A_\mu = 0. \tag{40}$$

These formulas look the same as for PCM's, but A_μ is given in terms of $g(x)$ by a completely different formula (eqs. (37) and (38) instead of eq. (2).

In two dimensions we once again have the Bianchi identity $F_{+-} = 0$ and the equation of motion $\partial_+ A_- + \partial_- A_+ = 0$. Therefore, it is natural to investigate whether the formulas that gave rise to symmetries of PCM's also gives rise to symmetries in this case. With this motivation, we once again form the Lax pair of equations

$$(\partial_\pm + \alpha_\pm A_\pm)X = 0, \tag{41}$$

and note that they are compatible if we write $\alpha_\pm$ in terms of a spectral parameter as in eq. (13). Then the solution is given by the contour independent integral

$$X(t) = P\exp\Big(-\int_{x_0}^{x}(\alpha_+ A_+ dy^+ + \alpha_- A_- dy^-)\Big), \tag{42}$$

as before. The obvious guess is that, just as for PCM's, the hidden symmetry is described by

$$\delta g = gX(t)\epsilon X(t)^{-1}. \tag{43}$$

This turns out to be correct. Under an arbitrary infinitesimal variation $g^{-1}\delta g = \eta(x) \in \mathcal{G}$, we have

$$\delta M = \eta^\dagger M + M\eta, \tag{44}$$

which implies that

$$\delta A_\mu = D_\mu \eta + D_\mu(M^{-1}\eta^\dagger M). \tag{45}$$

The first term is the same as for a PCM, but the second one is new. The symmetry requires that $\partial^\mu(\delta A_\mu) = 0$, when we substitute $\eta = X\epsilon X^{-1}$. The vanishing of the contribution from the first term in eq. (45) is identical to the PCM case. The second term in eq. (45) also has a vanishing divergence (for $\eta = X\epsilon X^{-1}$).

Next, we wish to study the algebra of these symmetry transformations. The commutator turns out to be

$$[\delta(\epsilon_1, t_1), \delta(\epsilon_2, t_2)]g = \frac{t_1\delta(\epsilon_{12}, t_1) - t_2\delta(\epsilon_{12}, t_2)}{t_1 - t_2} g + \delta' g + \delta'' g, \tag{46}$$

where the first term is the same as we found for PCM's, but there are two additional pieces. The $\delta' g$ term is a local $\mathcal{H}$ transformation of the form $h(x)g$, which is a symmetry of the theory. It is trivial in its action on $M = g^\dagger g$, which is all that appears in $\mathcal{L}$. (This is analogous to the trivial invariance of the Einstein–Hilbert action under local Lorentz transformations.) The $\delta'' g$ term is given by

$$\delta'' g = \frac{t_1 t_2}{1 - t_1 t_2} \left(\delta(\epsilon'_{12}, t_2) - \delta(\epsilon'_{21}, t_1)\right), \tag{47}$$

where

$$\epsilon'_{12} = M_0^{-1} \epsilon_1^\dagger M_0 \epsilon_2 - \epsilon_2 M_0^{-1} \epsilon_1^\dagger M_0, \tag{48}$$

and $M_0 = M(x_0)$.

As in the PCM, we define modes by contour integrals of the form given in eq. (21), and associate charges J_n^i to the transformation $\Delta_n(\epsilon)$. These can be converted to "currents" $J_n^i(\sigma) = \sum e^{in\sigma} J_n^i$. In the case of an SSM, there are two distinct classes of currents, those belonging to $\mathcal{H}$ and those belonging to $\mathcal{K}$. As Ref. [5] shows in detail, the significance of the $\delta'' g$ term in eq. (46) is that the $\mathcal{H}$ currents satisfy Neumann boundary conditions at the ends of the interval $0 \leq \sigma \leq \pi$, while the $\mathcal{K}$ currents satisfy Dirichlet boundary conditions at the two ends

$$J^{i\prime}(0) = J^{i\prime}(\pi) = 0 \quad \text{for} \quad J^i \in \mathcal{H} \tag{49a}$$

$$J^i(0) = J^i(\pi) = 0 \quad \text{for} \quad J^i \in \mathcal{K}. \tag{49b}$$

As a result, $J_n^i = J_{-n}^i$ for $\mathcal{H}$ charges and $J_n^i = -J_{-n}^i$ for $\mathcal{K}$ charges. Thus the mode expansions become

$$J^i(\sigma) = J_0^i + 2\sum_{n=1}^{\infty} \cos n\sigma J_n^i \quad \text{for} \quad J^i \in \mathcal{H} \tag{50a}$$

$$J^i(\sigma) = 2i \sum_{n=1}^{\infty} \sin n\sigma J_n^i \quad \text{for} \quad J^i \in \mathcal{K}. \tag{50b}$$

In terms of the modes, local current algebra on the line segment $0 \leq \sigma \leq \pi$ then implies that

$$[J_m^i, J_n^j] = f^{ij}{}_k (J_{m+n}^k + J_{m-n}^k) \quad \text{for} \quad J_n^j \in \mathcal{H} \tag{51a}$$

$$[J^i_m, J^j_n] = f^{ij}{}_k(J^k_{m+n} - J^k_{m-n}) \quad \text{for} \quad J^j_n \in \mathcal{K}. \tag{51b}$$

I propose to call this kind of a current algebra $\hat{G}_H$.

This result is somewhat surprising, because it seems to conflict with claims in the literature that the symmetry is an ordinary $\hat{G}$ current algebra on a circle, like what we found for PCM's. Actually, a few authors did previously obtain the same $\hat{G}_H$ algebra for a special class of SSM's, though they did not offer an interpretation[15]. As an additional check on the result, Ref. [5] studies the formulation of PCM's as SSM's based on the coset $G \times G/G$ [16]. It shows that the $\hat{G}_H$ symmetry of this model agrees with $\hat{G}$, the symmetry of the PCM. If the symmetry of a G/H SSM were a full $\hat{G}$ (rather than the subgroup $\hat{G}_H$), then the SSM construction of a PCM would imply that it has a $\hat{G} \times \hat{G}$ symmetry. Such symmetries occur for WZNW models, of course, but there are no Wess–Zumino terms in our models.

The Virasoro-like symmetries of PCM's also generalize to SSM's. The natural guess is that, just as for the current algebra symmetry, the same formula will describe the symmetry in this case, namely

$$\delta^V(t)g = g\Big((t^2-1)\dot{X}(t)X(t)^{-1} + I\Big). \tag{52}$$

This turns out to be correct, but once again the algebra differs from that of PCM's. We find that

$$[\delta^V(t_1), \delta(\epsilon, t_2)]g = \delta g + \delta' g + \delta'' g, \tag{53}$$

where δg is the PCM result given in eq. (27). The $\delta' g$ is a local $\mathcal{H}$ transformation and $\delta'' g$ contains new terms. (The formulas are given in Ref. [5].) The crucial question becomes what $\delta'' g$ contributes $[\delta^V_m, \delta_n(\epsilon)]g$, when we insert it into the appropriate contour integrals, or, equivalently, what it contributes to $[K_m, J^i_n]$. The result is

$$[K_m, J^i_n] = n(J^i_{m+n-1} - J^i_{m+n+1} - J^i_{n-m+1} + J^i_{n-m-1}). \tag{54}$$

The first two terms are the PCM result of eq. (29), while the last two terms are the new contribution arising from $\delta'' g$.

After our experience with the current algebra symmetry, the interpretation of the result (54) is evident. The generators K_m satisfy the restrictions $K_m = K_{-m}$, just like the $\mathcal{H}$ currents. In other words, $K(\sigma)$ satisfies Neumann boundary conditions at the ends of the interval $0 \leq \sigma \leq \pi$. Just as for PCM's, one can define a stress tensor

$$T(\sigma) = \frac{i}{2}\frac{K(\sigma)}{\sin\sigma}, \tag{55}$$

which satisfies the standard stress tensor algebra. As before, it is singular at $\sigma = 0$ and $\sigma = \pi$, so that modes L_m do not exist.

4 Concluding Remarks

We have seen that the key ingredient in the study of hidden symmetries of two-dimensional integrable models is the group element obtained by integrating the Lax pair

$$X(x,t) = P\exp\Big\{ -\int_{x_0}^{x} (\alpha_+ A_+ dy^+ + \alpha_- A_- dy^-)\Big\}. \tag{56}$$

Since it plays such a central role, it is natural to explore what happens if one makes a change of variables

$$g'(x) = g(x)X(x,u), \quad -1 < u < 1. \tag{57}$$

The result is quite different for PCM's and SSM's. In the case of PCM's, it turns out the g' equation of motion is

$$(1-u)\partial_+ A'_- + (1+u)\partial_- A'_+ = 0. \tag{58}$$

This is recognized to be the equation of motion obtained from the action

$$S_u(g') = S_{PCM}(g') + uS_{WZ}(g'), \tag{59}$$

where S_{WZ} denotes a Wess–Zumino term. Thus we learn that all values of u other than $u = \pm 1$ give equivalent classical theories. The special values $u = \pm 1$, which are different, correspond to WZNW theory. For the quantum theory the normalization matters, and one should consider

$$S_{k,u} = k\Big(\frac{1}{u} S_{PCM} + S_{WZ}\Big), \tag{60}$$

where k is an integer. It is plausible that for a given k, all values of u other than ± 1 give equivalent quantum theories.

In the case of SSM's the change of variables in eq. (57) does not give rise to a WZ term. Instead it is a (finite) symmetry transformation that corresponds to exponentiating the infinitesimal symmetry generated by K_0.

There is much work that still remains to be done if the analysis presented here is going to be applied to the study of string theory duality symmetries. Obvious future directions include coupling the models to 2D gravity as well as adding fermions and supersymmetrizing. Other issues involve understanding how quantization breaks the symmetry to discrete subgroups. This requires dealing with finite symmetry elements, rather than the infinitesimal elements described here. Considerable progress in addressing these issues has been made by Sen in his study of the toroidally compactified

heterotic string in two dimensions[13]. He identified discrete current algebra symmetries. It would be interesting to determine whether there are also discrete Virasoro symmetries.

Since compactification to one dimension is expected to give even larger hyperbolic Lie algebra symmetry groups, that should be very interesting to explore. A first step, it seems to me, would be to understand how the two-dimensional analysis is modified when the spatial dimension is a circle. As we have pointed out already, the formula for X in eq. (14) is no longer single-valued in that case, so new issues arise.

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